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TITLE- Optimization of a Very Low Capacity Channel Using a Multi-Tone Frequency Shift Keyed Detector

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AUTHOR(S)- L. Schuchman

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ABSTRACT

In several low capacity space communication channels, oscillator instabilities have to be accounted for in the design of a communication system. Planetary and lunar sensor networks, and deep space probes communicating with Earth are two such low capacity channels.

An optimum maximum likelihood detector is derived for a signal which has been distorted by phase noise caused by transmitter oscillator instabilities. The phase noise model used is believed to be a conservative model in that it assumes very little is known as to the noise structure. The detector differs in design from that believed to be optimum for very low capacity channels (independent of the phase noise model used). This belief is based on the assumption that signals designed for low capacity channels are transmitted at low signal-to-noise predetection ratios. This in turn leads to concern with the definition of low signal-to-noise ratios. Finally it is shown that for the derived detector reasonable operation [error rate performance of 10^{-2} or less] does not result in low signal-to-noise predetection ratios.

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TECHNICAL MEMORANDUM

At low data rates and for very low capacity channels, the effects of oscillator instabilities cannot be ignored. That is, if an oscillator is designed so that the frequency is accurate to within one part in 10^8 on a short term basis and if we are transmitting at a desired carrier frequency of 2×10^9 hertz, then the carrier frequency can vary by as much as 20 hertz.

There are several space channels for which such carrier frequency deviations can result in serious problems. For example, it has been suggested that in lunar explorations a network of low power - low data rate sensors be distributed on the lunar surface. Since the temperature control of the several oscillators generating RF carriers (at S-band) would be taxed by the wide temperature variations on the Moon, such a network design would have to account for oscillator instability. In addition, oscillator instabilities are of concern in the transmission of data from low power deep space probes. In this paper we mathematically model the phase instability and then derive the optimum detector for this very low capacity channel.*

Several researchers [1,2,3] have devised models which attempt to mathematically characterize the oscillator randomness. In particular, Viterbi has modeled the oscillator randomness in terms of transmissions of narrow band Gaussian processes centered around some known frequency ω_o. This means that the transmitted signal has a uniformly distributed phase uncertainty together with a Raleigh fading amplitude. Fergurson[2,3] points out that this model suffers in that most researchers believe that amplitude variations due to oscillator instabilities are not significant. Ferguson then goes on to describe several models in each of which the variations of the oscillator is characterized as a random phase process. He then shows that no matter how one characterizes the oscillator randomness the derived optimum incoherent detector asymptotically reduces to a quadratic processor for received signals with small predetection signal-to-noise ratios. The quadratic detector is one which has the following form for low signal-to-noise ratios:

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*The term optimum is used in the Bayesean sense of minimizing the average risk or equivalently deriving the maximum likelihood detector for the transmission of equi-likely signals.

$$\int_0^T \int_0^T y(t)\ y^*(u)\ R(t,u)dt\ du,$$

where

$y(t)$ is the received signal,

$y^*(u)$ is the complex conjugate of $y(t)$, and

$R(t,u)$ is the weighting function that must satisfy the properties of an auto-correlation function.

Finally Ferguson describes and evaluates the performance of a very good realization of a quadratic processor, the spectrum analyzer.

This paper is motivated by the realization that the low signal-to-noise model may not be reasonable for many very low capacity channel applications. Thus, we will show how a particular model of the oscillator phase randomness leads to a realizable optimum detector for <u>all</u> signal-to-noise ratios and in the process we will show why the quadratic detector may not be valid for its prescribed application.

Both Viterbi and Ferguson assumed the use of an M'ary orthogonal transmission alphabet to provide a coding improvement at the expense of an increased information bandwidth. We will do the same in this paper and in addition it is assumed that our orthogonality is achieved by proper frequency separation.* We thus transmit one of M equally possible frequency orthogonal messages in a time T containing $\log_2 M$ bits of information. The receiver has to determine which of the M signals is transmitted when the transmission is distorted by the phase noise $\theta_i(t)$ and additive white Gaussian noise. $\theta_i(t)$ is a stepped approximation to the actual phase noise and is illustrated in Figure 1.

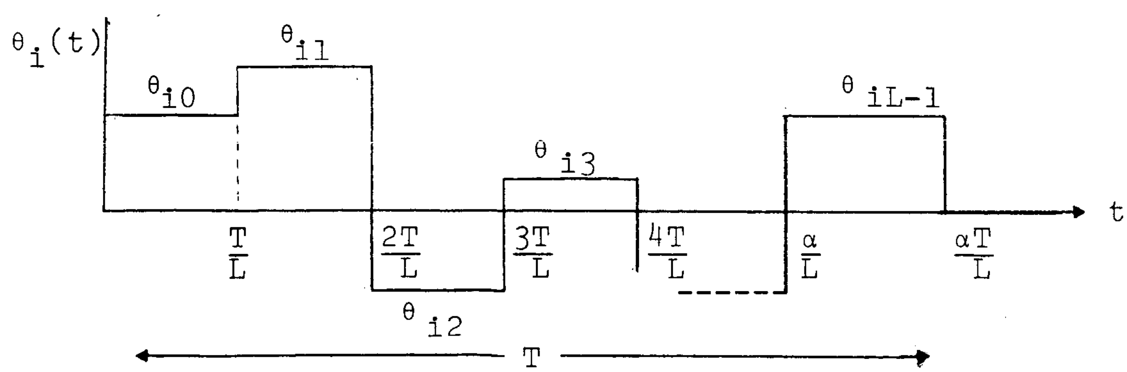

FIGURE 1: The Random Phase Noise As a Function of Time

*The orthogonality of time is realized by orthogonal frequency separations in MFSK.

Mathematically $\theta_i(t)$ is described as follows:

$$\theta_i(t) = \sum_{\alpha=0}^{L-1} \theta_{i\alpha} \left[u[t - \frac{\alpha}{L} T] - u[t - \frac{\alpha+1}{L} T]\right] \tag{1}$$

where

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

The variate $\theta_{i\alpha}$ is independent and uniformly distributed from $-\Pi$ to Π for all α and i.

$$i \in \{1,2,\cdots M\}$$

The transmitted signal x(t) can therefore be written as

$$x(t) = \left(\frac{2E}{T}\right)^{1/2} k \cos(\omega_i t + \theta_i(t)) \tag{2}$$

where E is the energy in the received symbol of duration T. The parameter k is the propagation loss factor (k>1).

The received signal y(t) is then given by

$$y(t) = \frac{x(t)}{k} + \sum_{i=1}^{M} [n_{ij}(t) \cos \omega_i t - n_{2i}(t) \sin \omega_i t] \tag{3}$$

where $\{n_{ij}(t)\}$ and $\{n_{2i}(t)\}$ are white, Gaussian, and independent random variables.

In Appendix A the optimum maximum likelihood detector is derived with the result that the detector computes M energy measures $\gamma_k(\{z_{jk}\})$ and decides that the signal transmitted is

$$\max \left\{ \gamma_k(\{z_{jk}\}) : \quad k = 1,2,\cdots M \right\} \tag{4}$$

where $\gamma_k(\{z_{jk}\})$ is given by

$$\gamma_k(\{z_{jk}\}) = \begin{cases} \sum_{j=0}^{L-1} f(z_{jk}) + \ln \prod_{j=1}^{L-1} g(z_{jk}) & L \geq 2 \\ z_{jk} & L = 1 \end{cases} \tag{5}$$

where

$$f(z_{jk}) = \begin{cases} 0 & z_{jk} \leq 3.75 \\ z_{jk} & z_{jk} > 3.75 \end{cases} \tag{6}$$

$$g(z_{jk}) \cong \begin{cases} h(z_{jk}) & z_{jk} \leq 3.75 \\ \dfrac{K(z_{jk})}{z_{jk}^{1/2}} & z_{jk} > 3.75 \end{cases} \tag{7}*$$

$$h(z_{jk}) = 1 + 3.5156229\eta^2 + 3.0899424\eta^4 + 1.2067492\eta^6 + 2.659732\eta^8 + .0360768\eta^{10} + .0045813\eta^{12} \tag{8}$$

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*Where the error in the approximation is less than 12×10^{-7}.

$$K(z_{jk}) = .39894228 + .01328592\eta^1 + .00225319\eta^1 - .00157565\eta^1 + .00916281\eta^1 = .02057706\eta^1 + .02635537\eta^1 - .01647633\eta^1 + .00392377\eta^1 \qquad (9)$$

where $\eta = z_{jk}/3.75$.

The z_{jk}'s are determined by computing the received band-pass envelope correlation energy in each of the M sub-channels {k: 1,2,$\cdots$M} for each of the L time intervals in which the phase θ_{jk} is constant {j: The correlation period is from $j\frac{T}{L}$ to $\frac{(j+1)T}{L}$, with $0 \leq j \leq L-1$}.

It is shown in Appendix B that a very good, although less than optimum, decision rule for this system is simply to compute the following $\tilde{\gamma}_k(\{z_{jk}\})$ energy measures and select the max $\tilde{\gamma}_k(\{z_{jk}\})$ where

$$\tilde{\gamma}_k(\{z_{jk}\}) = \begin{cases} \sum_{j=0}^{L-1} z_{jk}^2 \text{ if max } z_{jk} \text{ of max } \tilde{\gamma}_k(\{z_{jk}\}) < C \\ \sum_{j=0}^{L-1} V(z_{jk}) \qquad \text{otherwise} \end{cases} \qquad (10)$$

where

$$V(z_{jk}) = \begin{cases} C & z_{jk} \leq C \\ z_{jk} & z_{jk} > C \end{cases}$$

C is optimized by a trial and error process and is a function of L. As is shown in Appendix B, C is in the neighborhood of the number 1.

We return now to the question of the validity of the small signal model. If the quadratic detector is to be optimum than all $(z_{jk})^{\ell}$ terms in the polynomial expansion for $h(z_{jk})$ (given by Equation 8) for which ℓ is greater than 2 must be insignificant. To show that this is not the case we note that z_{jk} is a Rician distributed random variable whose density distribution is given by

$$p(z_{jk}) = \frac{z_{jk}}{2\left(\frac{E}{N_o}\right)\frac{1}{L}} \exp\left[\frac{-\left[z_{jk}^2 + \left(\frac{2E}{N_oL}\right)^2\right]}{2 \cdot \frac{2E}{N_oL}}\right] I_o(z_{jk})^* \tag{11}$$

The expectation of z_{jk}^{ℓ} is given by[4]

$$E(z_{jk}/k \text{ transmitted}) = \left(\frac{E}{N_oL}\right)^{\ell/2} 4^{\ell/2}\, \Gamma(1/2\ell+1)$$

$$\cdot \; {}_1F_1\left(\frac{1}{2}\ell+1;\; 1;\; \left(\frac{E}{N_oL}\right)\right)\left[\exp^{-\frac{E}{N_oL}}\right]$$

$$E(z_{jk}^{\ell}/i(i\neq k) \text{ transmitted}) = \left(\frac{E}{N_oL}\right)^{\ell/2} 4^{\ell/2}\, \Gamma(1/2\ell+1) \tag{12}$$

where $\Gamma(\alpha)$ is the gamma function and ${}_1F_1(a;b;c)$ is a confluent hypergeometric function.

The expectation of z_{jk}^2 and z_{jk}^4 has been computed with the aid of Janke and Emde[5] and the results are plotted in Figure 2.

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*L is equal to the product of the frequency uncertain bandwidth times the transmitted symbol period (T).

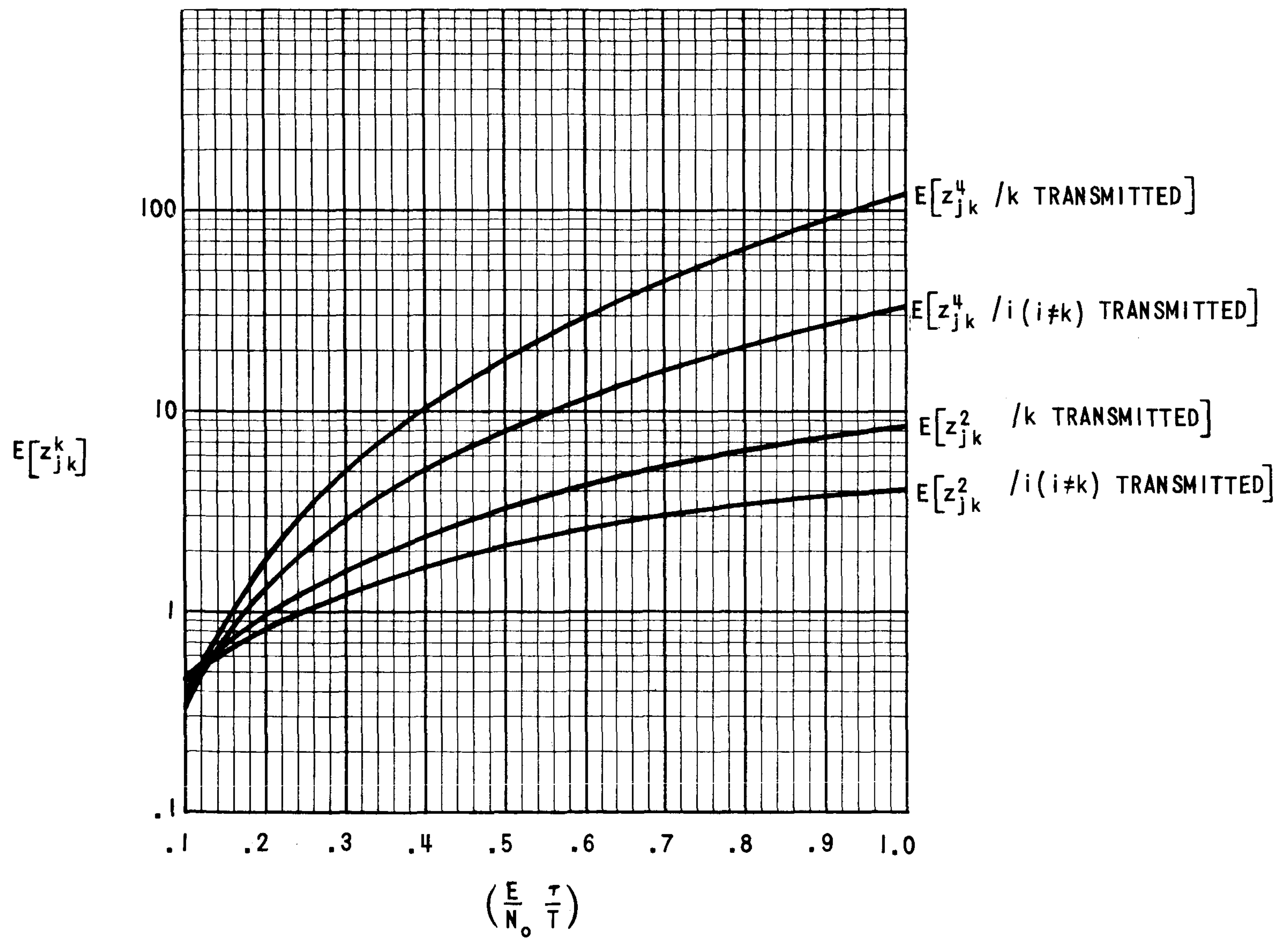

FIGURE 2 - THE EXPECTATION OF z^2_{jk} AND z^4_{j1} AS A FUNCTION OF $\left(\frac{E}{N_o}\;\frac{\tau}{T}\right)$.

It is to be noted that for values of $\frac{E}{LN_o}$ as small as .2 the expectation of z_{jk} is greater than 1.* Thus, on the average, the z_{jk} terms, in Equation 9, of the 4^{th} order will not be insignificant for values of $\frac{E}{LN_o}$ greater than .2. Since the energy per word-to-noise spectral density can be interpreted as the designed minimum value for acceptable performance we find the following. To achieve a bit error rate of better than 10^{-2} requires an E/N_o of at least 10 for optimum detection of non-coherent FSK. Thus, for $L \leq 50$ the optimum detector could not be realized as a quadrature processor. Since the required predetection energy-to-noise spectral density for the MFSK system goes up for a given value of bit error rate, we can conclude that for all MFSK systems, the optimum receiver differs from that of the quadrature processor for values of $L \leq 50$.*

Finally we note that in more conventional receivers the absolute values of the energy measures are not significant but only their ratios are. However, in the decision rules given by Equations (4) and (10) the absolute values of the z_{jk} parameters are significant and are given by Equation (A-5) for optimum performance. This means that the optimum receiver must be able to estimate the received word energy E and word energy-to-noise spectral density E/N_o.

Results

It has been shown that a realization of an optimum detector for a reasonable model of an MFSK time varying phase uncertain transmission can be achieved. Unfortunately, the calculation of the error rate metric, does not lend itself to analysis but does to a simulation or experimental investigation.

It would thus be interesting to determine the performance of this receiver and compare it to that of the spectrum analyzer. In addition it is believed that the 1.9×10^{-7} error in $g(z_{jk})$ Equation (9) is probably too rigid. Thus, one would

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*It is to be noted that in Ferguson's paper[3] the graphical results are for values of L and E/N_o which do not satisfy the small signal optimality criterion.

expect that $h(z_{jk})$ and $K(z_{jk})$, Equations (8) and (9), respectively, can be simplified significantly without a correspondingly significant loss in detector performance. This too can be determined experimentally.

Acknowledgment

The author wishes to thank Mr. Robert L. Selden for several very helpful discussions.

2034-LS-gdn

L. Schuchman

Attachments
References
Appendix A - Derivation of the Optimum Maximum Likelihood Detector
Appendix B - A Simple Near-Optimum Decision Rule

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APPENDIX A

DERIVATION OF THE OPTIMUM MAXIMUM LIKELIHOOD DETECTOR

We assume the signal transmitted is as described by Equation (2) and the received signal is given by Equation (3). The maximum likelihood detector for this M'ary FSK transmission then makes M-1 likelihood comparisons to determine the most likely signal transmitted. Each of the likelihood ratios $\ell(\omega_\ell/\omega_h)$ is computed in the following manner

$$\ell\left(\frac{\omega_\ell}{\omega_h}\right) = \frac{P_r[y(t)/\omega_\ell]}{P_r[y(t)/\omega_h]} \tag{A-1}$$

Where $P_r[y(t)/\omega_\alpha]$ is the probability density of y(t) assuming that signal α was transmitted.

This probability can be written conditionally as

$$P_r[y(t)/\omega_\alpha] = \int_{\{\theta_i\}} P_r(y(t)/\omega_\alpha,\{\theta_i\})\ P(\{\theta_i\})\ d\{\theta_i\} \tag{A-2}$$

where $\{\theta_i\}$ is the set of independent uniformly distributed phase variables defined by Equation (1).

$P(\{\theta_i\})$ is the joint probability density of $\{\theta_i\}$.

Substituting Equation (A-2) into Equation (A-1) and making use of the narrow band, white, and Gaussian assumption about the noise leads to

$$\ell\left(\frac{\omega_\ell}{\omega_h}\right) = \frac{\dfrac{1}{(2\pi)^L}\displaystyle\iiint_{-\pi}^{\pi} e^{-\left[\sum_{j=1}^{L}\sum_{i=1}^{2\tau W}\left(y_{ij}-\sqrt{\frac{2E}{T}}\cos\left[\omega_\ell\left[\frac{i}{2W}+(j-1)\tau\right]+\theta_{\ell j}\right]\right)^2/2\sigma_N^2\right]} d\theta_{\ell 1}\, d\theta_{\ell 2}\cdots d\theta_{\ell L}}{\dfrac{1}{(2\pi)^L}\displaystyle\iiint_{-\pi}^{\pi} e^{\left[-\sum_{j=1}^{L}\sum_{i=1}^{2\tau W}\left(y_{ij}-\sqrt{\frac{2E}{T}}\cos\left[\omega_h\left[\frac{i+(j-1)\tau}{2W}\right]+\theta_{hj}\right]\right)^2/2\sigma_N^2\right]} d\theta_{h1}\, d\theta_{h2}\cdots d\theta_{hL}}$$

(A-3)

where $$\tau = \frac{T}{L}$$

Using the narrow band assumption Equation (A-3) can be written as

$$\ell\left(\frac{\omega_\ell}{\omega_h}\right) = \frac{\prod_{j=1}^{L} \int_{-\pi}^{\pi} e^{-\sum_{i=1}^{2\tau W} y_{ij}\sqrt{\frac{2E}{T}} \cos\left(\omega_\ell\left[\frac{i}{2W} + (j-1)\tau\right) + \theta_{\ell j}\right]/\sigma_N^2} d\theta_{\ell j}}{\prod_{j=1}^{L} \int_{-\pi}^{\pi} e^{-\sum_{i=1}^{2\tau W} y_{ij}\sqrt{\frac{2E}{T}} \cos\left(\omega_h\left[\frac{i}{2W} + (j-1)\tau\right) + \theta_{hj}\right]/\sigma_N^2} d\theta_{hj}}$$

which is equivalently

$$\ell\left(\frac{\omega_\ell}{\omega_h}\right) = \frac{\prod_{j=1}^{L} I_o\left(\frac{2}{N_o}\left(\left[\int_{(j-1)\tau}^{j\tau} y(t)\sqrt{\frac{2E}{T}} \cos \omega_\ell t \, dt\right]^2 + \left[\int_{(j-1)\tau}^{j\tau} y(t)\sqrt{\frac{2E}{T}} \sin \omega_\ell t \, dt\right]^2\right)^{\frac{1}{2}}\right)}{\prod_{j=1}^{L} I_o\left(\frac{2}{N_o}\left(\left[\int_{(j-1)\tau}^{j\tau} y(t)\sqrt{\frac{2E}{T}} \cos \omega_h t \, dt\right]^2 + \left[\int_{(j-1)\tau}^{j\tau} y(t)\sqrt{\frac{2E}{T}} \sin \omega_h t \, dt\right]^2\right)^{\frac{1}{2}}\right)} \tag{A-4}$$

where the standard approximation

$$\frac{1}{2W} \sum_{i=1}^{2\alpha W} x_i z_i \cong \int_0^{\alpha} x_i z_i \, dt$$

has been used.

$I_o(\beta)$ is the modified Bessel function of the first kind and zero order with argument β.

Thus we see from Equation (A-4) that the arguments of the modified Bessel functions are simply the narrow band envelope outputs of a non-coherent matched filter, where the integration time τ is the time in which the random phase is constant. As a simple check we note that for L=1 the receiver reduces to one well known for the optimum non-coherent detection of MFSK.

The form of the detector as implied by Equation (A-4) has been previously derived by Ferguson. However, Ferguson[3] concluded that such a receiver would be to difficult to realize. However, if we made use of the following polynomial expansions,[6]

$$x \le 3.75$$

$$I_o(x) = 1 + 3.5156229t^2 + 3.0899424t^4 + 1.2067492t^6 + 2.659732t^8 + .0360768t^{10} + .0045813t^{12} + \varepsilon$$

$$|\varepsilon| < 1.6 \times 10^{-7}$$

$$(A-5)^4$$

$$3.75 \le x$$

$$x^{\frac{1}{2}}e^{-x}I_o(x) = .39894228 \times .03988024t^{-1} - .00362018t^{-2} + .00163801t^{-3} = .01031555t^{-4} + .02282967t^{-5} .02895312t^{-6} + .01787654t^{-7} - .00420059t^{-8} + \varepsilon$$

$$|\varepsilon| < 2.2 \times 10^{-7}$$

then the decision criteria can be given as in Equations 3 and 4 of the text.

APPENDIX B

A SIMPLE NEAR-OPTIMUM DECISION RULE

As discussed in Appendix A the optimum receiver forms M weighted measures

$$\prod_{k=o}^{L-1} I_o(a_k) \qquad \text{where} \qquad a_n > 0$$

and compares them to determine which is maximum. Let us first take the case for L=2. We then may write

$$\prod_{k=o}^{1} I_o(a_k) = \frac{1}{\pi 2} \int_o^{\pi}\int_o^{\pi} e^{\pm a_1 \cos\theta_1 \pm a_2 \cos\theta_2} d\theta_1 d\theta_2 \qquad \text{(B-1)}$$

If we expand the exponentials in an infinite series

$$\prod_{k=o}^{1} I_o(a_k) = \frac{1}{\pi 2} \int_o^{\pi}\int_o^{\pi} \sum_{j=0}^{\infty} \frac{(\pm a_1 \cos\theta_1 \pm a_2 \cos\theta_2)^j}{j!} d\theta_1 d\theta_2$$

which can be written as

$$\prod_{k=o}^{1} I_o(a_k) = \frac{1}{\pi 2} \int_o^{\pi}\int_o^{\pi} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \binom{j}{m} (\pm a_1 \cos\theta_1)^m (\pm a_2 \cos\theta_2)^{j-m} d\theta_1 d\theta_2 \qquad \text{(B-2)}$$

Using the following identities

$$[\cos\theta]^{\alpha} = \left(\frac{1}{2}\right)^{\alpha} e^{j\theta\alpha}[1+e^{-2j\theta}]^{\alpha}$$

$$[\cos\theta]^{\alpha} = \left(\frac{1}{2}\right)^{\alpha} \sum_{t=0}^{\alpha} e^{j\theta(\alpha-2t)} \binom{\alpha}{t} =$$

$$\left(\frac{1}{2}\right)^{\alpha-1} \sum_{t=0}^{\frac{\alpha-1}{2}} \binom{\alpha}{t} \cos(\alpha-2t)\theta \qquad \alpha \text{ odd}$$

$$\left(\frac{1}{2}\right)^{\alpha-1} \sum_{t=0}^{\frac{\alpha-1}{2}} \binom{\alpha}{t} \cos(\alpha-2t)\theta + \left(\frac{1}{2}\right)^{\alpha} \binom{\alpha}{\frac{\alpha}{2}} \qquad \alpha \text{ even} \tag{B-3}$$

Using Equation (B-3), Equation (B-2) reduces to

$$\prod_{k=o}^{1} I_o(a_k) = \sum_{\substack{j=0 \\ j \text{ even}}}^{\infty} \sum_{\substack{m=0 \\ m \text{ even}}}^{m} \frac{\binom{j}{m} \; m! \; (j-m)! \; a_1^m \, a_2^{j-m}}{j!\left[\left(\frac{j-m}{2}\right)! \; \left(\frac{m}{2}\right)!\right]^2 a^{j-m} \, 2^m} \tag{B-4}$$

which simplifies to

$$\prod_{k=o}^{1} I_o(a_k) = \sum_{\substack{j=0 \\ j \text{ even}}}^{\infty} \sum_{\substack{m=0 \\ m \text{ even}}}^{m} \frac{a_1^m \, a_2^{j-m}}{2^j\left[\left(\frac{m}{2}\right)! \; \left(\frac{j-m}{2}\right)!\right]^2} \tag{B-5}$$

Let $\frac{m}{2} = \alpha$ and $\frac{j}{2} = \beta$ then we can write Equation (B-5) as

$$\prod_{k=o}^{1} I_o(a_k) = \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\beta} \frac{a_1^{2\alpha} a_2^{2\beta}}{2^{2\beta}[(\alpha)!\ (\beta-\alpha)!]^2} \tag{B-6}$$

and finally

$$\prod_{k=o}^{1} I_o(a_k) = \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\beta} \frac{\binom{\beta}{\alpha}^2 a_1^{2\alpha}\ a_2^{2\beta}}{2^{2\beta}[\beta!]^2} \tag{B-7}$$

which can be bounded by

$$\begin{aligned} \prod_{k=o}^{1} I_o(a_k) &\leq \sum_{\beta=0}^{\infty} \frac{1}{2^{2\beta}[\beta!]^2} (a_1+a_2)^{2\beta} \\ &\geq \sum_{\beta=0}^{\infty} \frac{1}{2^{2\beta}[\beta!]^2} (a_1^2+a_2^2)^{\beta} \end{aligned} \tag{B-8}$$

and in general it can be shown that

$$\begin{aligned} \prod_{k=o}^{L-1} I_o(a_k) &\leq \sum_{\beta=0}^{\infty} \frac{1}{2^{2\beta}[\beta!]^2} \left[\left(\sum_{k=o}^{L-1} a_k\right)^2\right]^{\beta} \\ &\geq \sum_{\beta=0}^{\infty} \frac{1}{2^{2\beta}[\beta!]^2} \left[\sum_{k=o}^{L-1} a_k^2\right]^{\beta} \end{aligned} \tag{B-9}$$

Thus

$$\left(\sum_{k=o}^{L-1} a_k\right)^2 < \sum_{k=o}^{L-1} b_k^2 \Longrightarrow \prod_{k=o}^{L-1} I_o(a_k) \quad \prod_{k=o}^{L-1} I_o(b_k) \tag{B-10}$$

Unfortunately

$$\prod_{k=o}^{L-1} I_o(a_k) < \prod_{k=o}^{L-1} I_o(b_k)$$

does not necessarily imply

$$\sum_{k=o}^{L-1} a_k^2 < \sum_{k=o}^{L-1} b_k^2$$

By use of a trial and error method a decision rule, based on the bounds given in Equation (B-10), has been derived and is presented in Equation (10) of the text. To see how this rule compares with the optimum decision rule, Table B-1 is given* with T=1 and L=2. It is to be noted that the region uncertainty occurs only when

$$|\tilde{\gamma}_{k_1}(\{z_{jk_1}\}) - \tilde{\gamma}_{k_2}(\{z_{jk_2}\})| < |$$

It is believed that this region of uncertainty increases with L at a slow rate but this has not been verified.

*The modified Bessel function was evaluated using National of Bureau of Standards' Tables.[6,7]

TABLE B-1

Comparison of Optimum Decision Rule With The One Given by Equation (10) - L=2

x		y		Decision	
x_1	x_2	y_1	y_2	Equation (12) Choose	Optimum Choose
.7	.5	.9	.3	y	y
8.0	4.8	9.0	3.0	x	x
8.0	4.2	9.0	3.0	x	x
8.0	4.0	9.0	3.01	y	y
8.0	8.0	10.0	5.0	x	x
8.0	8.0	10.0	5.9	x	x
8.0	8.0	10.0	6.1	y	y
7.0	7.0	10.0	.1	x	x
6.0	4.9	10.0	.1	y	y
20.0	.1	10.0	10.0	x	x
20.0	.1	5.1	16.0	x*	y*
.8	.8	.9	.69	y	y
.8	.8	1.0	.5	y	y
1.01	5.01	.9	.9	x*	y*
3.0	4.0	5.0	1.9	x	y
6.0	.1	10.0	.1	x*	y*
6.0	5.45	10.0	.1	x*	y*
6.0	5.5	10.0	.1	x	x
1.3	.01	.9	.9	x	x
3.05	4.0	5.0	1.9	x	x
20.5	.1	5.5	16.0	y	y

*Asterisk identifies points where decision rules differ in decision.